Mathematics and Climate Seminar
Lorenz Equations

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The Lorenz Model

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = \rho x - y - xz \\ \dot{z} = -\beta x + xy \end{cases}$$

x is the spatial average of the hydrodynamic velocity

y is the temperature

z is the temperature gradient

σ is the Prandtl number

ρ is the Rayleigh number

$$\sigma, \rho, \beta > 0$$

$$\sigma > 1 + \beta$$

ρ is varied

The Lorenz Model

Definition

Chaotic dynamics is when the solution never repeats its past history exactly and moreover all approximate repetitions have finite duration.

Definition

Strange attractors are attractors that may contain very irregular orbits.

Properties of the Lorenz Equations

The Lorenz equations are symmetric with respect to the reflection

$$\psi : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} -x \\ -y \\ z \end{pmatrix}.$$

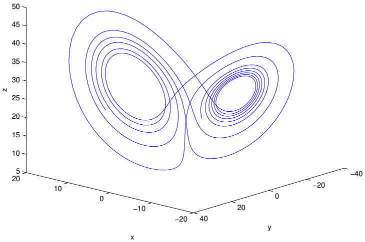

Figure: Plot of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

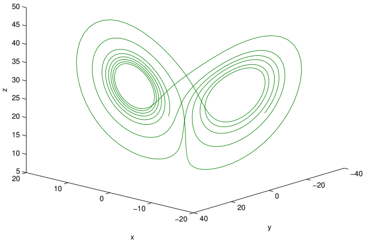

Figure: Plot of $\begin{pmatrix} -x \\ -y \\ z \end{pmatrix}$

Properties of the Lorenz Equations

The z-axis is an invariant set: if $x = y = 0$ at some point t, then $x = y = 0$ at all times. Moreover, any solution with $x = y = 0$ tends to zero as $t \to \infty$

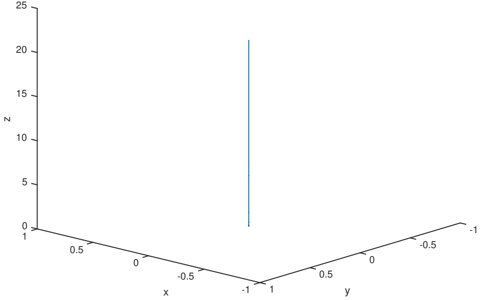

Figure: Plot of $\begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$

Properties of the Lorenz Equations

All solutions are eventually trapped in a bounded region of the state space.

Definition

A trapping set for a dynamical system in $\mathbb{R}^n$ is a closed connected set $D \subset \mathbb{R}^n$ which, for a finite time T, is invariant with respect to the flow, i.e. there exists a $T \geq 0$ such that $\phi_t(D) \subset D$ for all $t \geq T$.

Equilibrium solutions

The Lorenz system

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = \rho x - y - xz \\ \dot{z} = -\beta x + xy \end{cases}$$

has the critical points $(0, 0, 0)$ and

$$C_{\pm} = (\pm\sqrt{\beta(\rho - 1)}, \pm\sqrt{\beta(\rho - 1)}, \rho - 1)$$

Equilibrium solutions

The Jacobian of the Lorenz system is

$$Df(x, y, z) = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -\beta \end{pmatrix}$$

We will now take a look at the linearized systems in the neighborhood of the origin and $C_{\pm}$.

Numerical Experiments

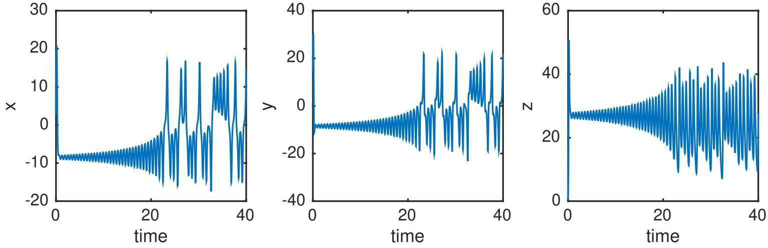

Figure: Solution when the Lorenz system is integrated with $\sigma = 10$, $\beta = \dfrac{8}{3}$, and $\rho = 28$, starting at the point $(3, 15, 1)$ for $0 \leq t \leq 40$.

Numerical Experiments

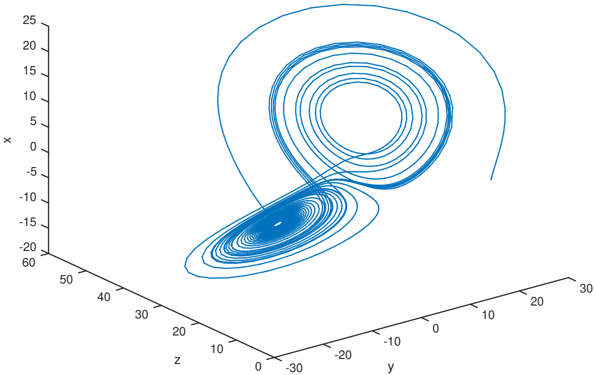

Figure: Orbit of the Lorenz equations emanating from $(3, 15, 1)$ for $\sigma = 10$, $\beta = \dfrac{8}{3}$, and $\rho = 28$.